ESN theorems for biunary semigroups

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We are interested in capturing the complete behaviour of classes of semigroups using categories obtained from them.

This means retaining only a small fraction of the products of pairs of elements.

The first case of this was the “ESN theorem” for inverse semigroups.

This approach can be applied more generally to so-called Ehresmann semigroups, as in [Lawson, 1991].

We’ll look at that case since it connects more directly to the generalisation we consider.

Algebraists have ways to define small categories without specifying any objects.

Instead one defines identities via “domain” and “range” operations acting on the arrows

and then one can add in the missing objects if one insists!

Here is one such way.

A small category C is a set with a partial binary operation $\circ$ and two unary operations denoted D and R, such that, for all $x, y \in C$:

1. $D(x) \circ x = x$, $x \circ R(x) = x$
2. $R(D(x)) = D(x)$, $D(R(x)) = R(x)$
3. $x \circ y$ exists if and only if $R(x) = D(y)$
4. if $R(x) = D(y)$ then $D(x \circ y) = D(x)$ and $R(x \circ y) = R(y)$
5. $x \circ (y \circ z) = (x \circ y) \circ z$ whenever the two products are defined.

It follows that $D(D(x)) = D(x)$, $R(R(x)) = R(x)$.

The identities of the category are $C^0 = \{D(x) \mid x \in C\}$

$= \{R(x) \mid x \in C\} = \{e \in C \mid D(e) = e\}$.

We can introduce an object for each identity, and recover the more usual definition involving objects.

But this or an equivalent purely algebraic definition is often used in algebra.

An *Ehresmann semigroup* $(S, \cdot, D, R)$ is a semigroup $(S, \cdot)$ with D, R unary, satisfying the following laws:

1. $D(x)x = x$ and $xR(x) = x$

2. $D(R(x)) = R(x)$ and $R(D(x)) = D(x)$

3. $D(x)^2 = D(x)$ (hence $R(x)^2 = R(x)$)

4. $D(xy) = D(xD(y))$, $R(xy) = R(R(x)y)$ (**congruence cond's**)

5. $D(x)D(y) = D(y)D(x)$

Again, it follows that $D(D(x)) = D(x)$, $R(R(x)) = R(x)$, and that

$$D(S) = \{D(s) \mid s \in S\} = \{R(s) \mid s \in S\} = \{e \in S \mid D(e) = e\}$$

is a subsemigroup which is a semilattice, the *projections* of S.

Every inverse semigroup is an Ehresmann semigroup if we define $D(s) = ss'$, $R(s) = s's$.

(This example is a *restriction semigroup*!)

More generally, the set of all binary relations on a set is an Ehresmann semigroup under relational composition,

with $D(s)$ the identity relation on $\mathrm{Dom}(s)$ and dually for $R(s)$.

Still more generally, examples arise from any small category:

take the elements to be sets of arrows and define domain, range and products in “the natural ways”.

(Actually, any small category becomes a special kind of Ehresmann semigroup if we add a zero element for undefined products!)

One can obtain a category from an Ehresmann semigroup $(S, \cdot, D, R)$ as follows:

define $x \circ y = xy$, but only when $R(x) = D(y)$

and keep the unary operations D and R.

Then $(S, \circ, D, R)$ is a category.

If we retain some further information from the Ehresmann semigroup, we can capture it entirely; see [Lawson, 1991].

Thus, if S is an Ehresmann semigroup, $D(S)$ is a semilattice under multiplication, hence is a poset with meets.

Extend this partial order in two ways to S:

define $s \leq_l t$ if $s = et$ for some $e \in D(S)$

or equivalently, $s \leq_l t$ if $s = D(s)t$.

This gives a partial order, and s is the unique $u \leq_l t$ such that $D(u) = D(s)$.

We define $\leq_r$ dually. Note that $\leq_l$ and $\leq_r$ agree on $D(S)$.

In fact we can totally describe the categories $C(S)$ we obtain in this way from Ehresmann semigroups S.

They each have two partial orders satisfying compatibility conditions

(assumed equal on C^0, which we assume is a meet-semilattice)

and then we can recover general semigroup products by assuming that "restriction" and "corestriction" is possible:

for $s \in C$ and $e \in C^0$ with $e \leq D(s)$, assume there is a unique $t \leq_l s$ for which $D(t) = e$; call it $e|s$, the *restriction* of s to e.

Define corestriction dually.

One can finitely describe bi-ordered categories $(C, \circ, D, R, \leq_l, \leq_r)$

that arise from Ehresmann categories in this way.

Call them Ehresmann categories.

Recover the "lost" semigroup operation by defining, for all $s, t \in C$.

$$s \otimes t = s|(R(s) \wedge D(t)) \ \circ \ (R(s) \wedge D(t))|t.$$

There is the hoped-for 1:1 correspondence, morphisms correspond, etc.

This approach generalise the proof of the original ESN theorem.

It has seen much generalisation to many settings where $D(S)$ is no longer a semilattice,

but perhaps a more general structure like a projection algebra,

sometimes one must abandon categories and move to some more general partial product

(that retains more of the original products),

and sometimes one must make do with only an equivalence of the two categories.

Lawson later realised that there was a more efficient way to define Ehresmann categories that does not require order; see [Lawson, 2021].

In an Ehresmann category, the restriction $e|s$ corresponds to the product es in the semigroup, but only when $e \leq D(s)$.

Instead, focus on algebraic properties of the left action of the semilattice $D(S)$ on S given by $e|s = es$ for *any* $e \in D(S)$, but forget about order.

Do the same for the obvious right action as well (and keep $\circ$, D, R)

and get a *category with Ehresmann biaction*.

So we retain information about products st where $R(s) = D(t)$

as well as arbitrary products es, se where $s \in S$ and $e \in D(S)$.

In particular we retain knowledge of ef where $e, f \in D(S)$.

In the Ehresmann or inverse cases, $ef \in D(S)$ and $D(S)$ is a semilattice.

So $e|s = (e \wedge D(s))|s$ and one can move freely between the biaction and the restriction/corestriction viewpoints if we know all projection meets.

But in some settings, restriction/corestriction is not available.

Lawson showed in [Lawson, 2021] that Ehresmann semigroups and categories with Ehresmann biaction are basically the same things.

Simultaneously, the authors of [Fitzgerald and Kinyon, 2021] took as their starting point *localisable semigroups*.

These satisfy all the Ehresmann semigroup laws except the commuting law for projections $D(s)D(t) = D(t)D(s)$,

and it is assumed that $D(S)$ is a subsemigroup (hence a band).

No orderings are available so they were forced to consider biactions.

Their version of categories with biaction was *transciption categories*.

So what actually is a transcription category?

It is a category with left and right actions of C^0 on C,

denoted by $e|s$ and $s|e$, and satisfying:

1. For $e, f \in C^0$, $e|f$ does not depend on interpretation.

2. For all $a \in C$, $D(a)|a = a$ and $a|R(a) = a$.

3. For all $a \in C$ and $e, f \in C^0$, $e|(f|a) = (e|f)|a$ and $a|(e|f) = (a|e)|f$. (These make sense because of a law to follow!)

4. For all $a, b \in C$, if $a \circ b$ exists then for all $e \in C^0$,

   (a) so does $(e|a) \circ R(e|a)|b$, and $e|(a \circ b) = (e|a) \circ R(e|a)|b$;

   (b) so does $a|D(b|e) \circ b|e$, and $(a \circ b)|e = a|D(b|e) \circ b|e$.

5. For all $e \in C^0$ and $a \in C$,

   (a) $D(e|a) = e|D(a)$;

   (b) $R(a|e) = R(a)|e$.

6. For all $e, f \in C^0$ and $a \in C$, $(e|a)|f = e|(a|f)$.

It follows that C^0 is a band under $ef := e|f$.

Every category with Ehresmann biaction is a transciption category:

just add the law $e|f = f|e$ for all $e, f \in C^0$.

(Lawson's laws look a little different but are equivalent to this case.)

Fitzgerald and Kinyon showed (similar to Lawson) that localisable semigroups and transcription categories are essentially the same things.

There are isomorphisms of suitable categories in both cases.

But the most important examples of localisable semigroups are Ehresmann semigroups...

is there a more general setting for this approach using biactions rather than order?

We'd like to be able to include examples where $D(S)$ need not even form a band!

There are lots of biunary semigroups like this.

Examples: *-regular semigroups (with $D(s) = ss'$, $R(s) = s's$),

hence regular $*$-semigroups (where $s^* = s'$),

and Baer $*$-semigroups, *e.g.* the mult. semigroups of Rickart *-rings.

These and Ehresmann semigroups are all *DRC-semigroups*, a variety of biunary semigroups considered from an ESN perspective in [Wang, 2022].

There are even some natural examples that fail to satisfy at least one of the congruence conditions

(e.g. binary relations with D, R and *demonic* composition).

Here's an absolute mimimum of laws if $(S, \cdot, D, R)$ is to become a category if we define $s \circ t = st$ whenever $R(s) = D(t)$:

1. $D(x)x = x$ and $xR(x) = x$

2. $D(R(x)) = R(x)$ and $R(D(x)) = D(x)$

3. $D(x)^2 = D(x)$ (hence $R(x)^2 = R(x)$)

Let's call such an S a *precat-semigroup*.

These form a variety of biunary semigroups.

So far, we have not been forced to assume the congruence conditions,

nor commutativity of projections in $D(S) = \{D(s) \mid s \in S\}$,

nor even that $D(S)$ forms a subsemigroup.

In a precat-semigroup, we define $s \circ t = st$ when $R(s) = D(t)$.

But $(S, \circ, D, R)$ still may not be a category!

It is a category exactly when the following laws are satisfied:

for all x, y, $R(x) = D(y) \Rightarrow (D(xy) = D(x) \;\&\; R(xy) = R(y))$.

Sufficient for these are the congruence conditions.

For then, if $R(x) = D(y)$, then

$D(xy) = D(xD(y)) = D(xR(x)) = D(x)$, and similarly $R(xy) = R(R(x)y)$.

Let's say a precat-semigroup $(S, \cdot, D, R)$ satisfying the above implications is a *cat-semigroup*.

The class of cat-semigroups turns out to be a proper quasivariety.

(There is a four-element cat-semigroup with a three-element quotient that is not a cat-semigroup.)

Cat-semigroups satisfying the congruence conditions form a variety,

since the cat-semigroup implicational laws become redundant.

For Ehresmann and localisable semigroups, the authors of [Lawson, 1991] and [Fitzgerald and Kinyon, 2021] defined

$s \circ t = st$ when $R(s) = D(t)$ for all $s, t \in S$, and

$e|s = es$ and $s|e = se$ for all $s \in S$ and $e \in D(S)$.

We can do exactly this for any cat-semigroup $(S, \cdot, D, R)$.

In the process we will get a category $\mathcal{C}(S) = (S, \circ, D, R)$ with some sort of “biaction”.

A category C equipped with a left and right action of C^0 on C, is said to be a *category with biaction* if it satisfies:

- for $e, f \in C^0$, $e|f$ does not depend on interpretation.
- for all $a \in C$, $D(a)|a = a$ and $a|R(a) = a$.
- for all $e, f \in C^0$ and $a \in C$, $(e|a)|f = e|(a|f)$.

(Laws 1, 2, and 6 for transcription categories.)

The category $\mathcal{C}(S)$ determined by the cat-semigroup S is a category with biaction if we define

$e|s = es$ and $s|e = se$ for all $s \in C$ and $e \in C^0$.

Can we retrieve S from $\mathcal{C}(S)$ equipped with this biaction?

This is impossible in principle:

there are two non-isomorphic four element cat-semigroups that determine isomorphic categories with biaction!

So we must narrow down our class if we want such a correspondence.

We also want to be able to describe the categories with biaction that arise from the semigroups.

So: we want a condition on the cat-semigroup $(S, \cdot, D, R)$ that

(a) allows one to go back and forth relatively easily, and

(b) includes all the extra examples we'd like to include.

For a localisable semigroup S, the semigroup operation can be recovered from the transcription category via

$$st = (s|D(t)) \circ (R(s)|t) \text{ for all } s, t \in S.$$

This works because:

1) for all $x, y \in S$, $R(xD(y)) = D(R(x)y)$, so the category product above exsts, and

2) $xy = xD(y)R(x)y$, so the category product recovers the semigroup product.

Proposition: The class of cat-semigroups satisfying the above two *strong match-up* laws is the variety of precat-semigroups satisfying

- the congruence conditions,
- the law $D(ef) = R(ef)$ (generalising $D(ef) = ef$), and
- the law $sD(t)R(s)t = st$.

This class properly contains the localisable semigroups.

Unfortunately there still aren't natural examples of cat-semigroups satisfying the strong match-up conditions...

other than Ehresmann semigroups!

So to admit more good examples...

we need a different pseudoproduct definition!

(For the previous one to work, we need the strong match-up conditions to hold.)

But in any cat-semigroup S, we can also write

$$st = (sD(t))(R(sD(t))t) = (sD(R(s)t))(R(s)t).$$

So if the cat-semigroup satisfies either

$$R(sD(t)) = D(R(sD(t))t) \text{ or } R(sD(R(s)t)) = D(R(s)t),$$

then we can write st as a category product in $\mathcal{C}(S)$:

$$st = s|D(t) \ \circ \ R(s|D(t))|t,$$

or else

$$st = s|D(R(s)|t) \ \circ \ R(s)|t.$$

We call these new laws

$$R(sD(t)) = D(R(sD(t))t), \; R(sD(R(s)t)) = D(R(s)t)$$

the *left and right match-up conditions* respectively;

either one is sufficient to allow recovery of S from $\mathcal{C}(S)$.

Strong match-up implies both left and right match-up...

but the converse certainly fails.

Proposition: The class of cat-semigroups satisfying the two match-up conditions is the variety of precat-semigroups satisfying

- $D(st) = D(sD(t))$ and $R(st) = R(R(s)t)$ (cong. cond's)

- $R(st) = D(R(st)R(t))$ and $D(st) = R(D(s)D(st))$.

Sufficient for the second pair of laws above are the laws

$R(st) = R(st)R(t) = R(t)R(st)$ and

$D(st) = D(s)D(st) = D(st)D(s)$.

The variety of precat-semigroups satisfying the congruence conditions plus

- $R(st) = R(st)R(t) = R(t)R(st)$ and

- $D(st) = D(s)D(st) = D(st)D(s)$

is the class of *DRC-semigroups*, mentioned before.

In [Wang, 2022], Shoufeng Wang obtained an ESN-style theorem for these using the more traditional order-based approach:

one has a projection algebra of identities, two partial orders in terms of which restriction and corestriction are defined.

However, a kind of generalised category notion is needed,

in which $s \circ t$ exists “more often” than just when $R(s) = D(t)$.

Our approach will apply to situations where no order is available

and we get actual categories.

What are the categories with biaction corresponding to cat-semigroups satisfying both match-up conditions?

Exactly those for which the derived algebra $(C, \otimes, D, R)$ satisfies all the right things!

This gives a finite axiomatisation of them, but is a “cop-out”.

In fact we can describe them more directly.

What are the categories with biaction corresponding to cat-semigroups satisfying both match-up conditions? Exactly those satisfying:

- $a|D(b) \ \circ \ R(a|D(b))|b$ exists and

  $e|(a|D(b) \ \circ \ R(a|D(b))|b) = e|a|D(b) \ \circ \ R(e|a|D(b))|b$;

- $a|D(R(a)|b) \ \circ \ R(a)|b$ exists and

  $(a|D(R(a)|b) \ \circ \ R(a)|b)|e = a|D(R(a)|b|e) \ \circ \ R(a)|b|e$.

These generalise the fourth pair of laws for transcription categories.

Given such a category with biaction, it becomes a cat-semigroup satisfying the match-up conditions if we define the pseudoproduct

$$s \otimes t = s|D(t) \;\circ\; R(s|D(t))|t.$$

But we get the same result if we define

$$s \otimes t = s|D(R(s)|t) \;\circ\; R(s)|t.$$

Is there a single symmetric equivalent definition?

Yes, but it needs three terms in the product!

$$s \otimes t = s|D(t) \;\circ\; R(s)|D(t) \;\circ\; R(s)|t.$$

How about those categories with biaction corresponding to cat-semigroups satisfying the strong matchup conditions?

For these, we simply add in the laws

1. $D(R(s)|t) = R(s|D(t))$ for all $s, t \in C$, and

2. $(e|f) \circ (e|f) = e|f$ for all $e, f \in C^0$.

(The product in the second law exists if we assume the first!)

Of course, transcription categories are special cases.

How about the categories when the precat-semigroups satisfy only the left match-up condition?

Proposition: The class of cat-semigroups satisying the left match-up condition is the variety of precat-semigroups satisfying

- $D(st) = D(sD(t))$ (left congruence)
- $R(xy) = R(R(xD(y))y))$ (right weak congruence)
- $R(st) = D(R(st)R(t))$

NOTE: the right congruence law does not follow!

But left conguence plus right weak congruence are sufficient to imply we have a cat-semigroup.

So what do the categories look like?

This is a little messy, but looks better in the following special case:

assume the precat-semigroup is *left semi-localisable*,

meaning that it satisfies the left congruence and right weak congruence conditions and $D(S)$ is a band.

The class of left semi-localisable semigroups includes the demonic relational composition example mentioned previously:

then we even have a left restriction semigroup $(S, \cdot, D)$,

but $(S, \cdot, R)$ only satisfies right weak congruence.

Also, every localisable semigroup is left semi-localisable.

And happily, the corresponding class of categories with biaction can be defined using a subset of the transcription category laws:

just make the fourth and fifth laws one-sided, giving

1. For $e, f \in C^0$, $e|f$ does not depend on how it's interpreted.

2. For all $a \in C$, $D(a)|a = a$ and $a|R(a) = a$.

3. For all $a \in C$ and $e, f \in C^0$, $e|(f|a) = (e|f)|a$ and $a|(e|f) = (a|e)|f$.

4. For all $a, b \in C$, if $a \circ b$ exists then for all $e \in C^0$, so does $(e|a) \circ R(e|a)|b$, and $e|(a \circ b) = (e|a) \circ R(e|a)|b$.

5. For all $e \in C^0$ and $a \in C$, $D(e|a) = e|D(a)$.

6. For all $e, f \in C^0$ and $a \in C$, $(e|a)|f = e|(a|f)$.

OPEN PROBLEM:

The left match-up condition came from noting that

$$st = (sD(t)) \cdot (R(sD(t))t),$$

in any cat-semigroup.

But we also have

$$st = (sD(t)D(R(sD(t))t)) \cdot R(sD(t))t),$$

which can be expressed in $C(S)$ and leads to a strictly more general left match-up condition...

OPEN PROBLEM:

Can we give a (finite?) first-order axiomatisation of the class of cat-semigroups which can be captured by their categories with biaction?

Is the class a variety/quasivariety?

REFERENCES

[Lawson, 1991]
M.V. Lawson, 'Semigroups and ordered categories I: the reduced case'. *J. Algebra* 141 (1991), 422–462.

[Lawson, 2021]
M. V. Lawson, 'On Ehresmann semigroups', *Semigroup Forum* 103 (2021), 953–965.

[Fitzgerald and Kinyon, 2021]
D. G. FitzGerald, M. K. Kinyon, 'Trace- and pseudo-products: restriction-like semigroups with a band of projections'. *Semigroup Forum* 103 (2021), 848–866.

[Wang, 2022]
S. F. Wang, 'An Ehresmann-Schein-Nambooripad type theorem for DRC-semigroups', *Semigroup Forum* 104 (2022), 731–757.

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Thanks!